Due Sun

8.4 – Matrices for General Linear Transformations

Recall this example from section 1.8

Find the standard matrix A for the linear transformation $T : R^2 \to R^3$ for which $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\-5\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\1\end{bmatrix}$ and use it to compute $T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)$.

And #3 from 4.7

Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \left\{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\right\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix}2\\1\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-1\\1\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix} \text{ and } \mathbf{u}_1' = \begin{bmatrix}3\\1\\-5\end{bmatrix}, \mathbf{u}_2' = \begin{bmatrix}1\\1\\-3\end{bmatrix}, \mathbf{u}_3' = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

a. Find the transition matrix B to B'.

b. Compute the coordinate vector $[\mathbf{w}]_B$, where $\mathbf{w} = \begin{bmatrix}-5\\8\\-5\end{bmatrix}$ and use the transition matrix in part (a) to compute $[\mathbf{w}]_{B'}$.

Now we combine these two concepts.

#5 Let $T : R^2 \to R^3$ be defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 + 2x_2\\-x_1\\0\end{bmatrix}$. (formula)

a. Find the matrix $[T]_{B',B}$ relative to the bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where

$$\overset{T(\vec{u}_1)}{\mathbf{u}_1 = \begin{bmatrix}1\\3\end{bmatrix}}, \overset{T(\vec{u}_2)}{\mathbf{u}_2 = \begin{bmatrix}-2\\4\end{bmatrix}}, \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$\left(\begin{array}{ccc|c|c} 1 & 2 & 3 & b_1 & c_1 \\ 1 & 2 & 0 & b_2 & c_2 \\ 1 & 0 & 0 & b_3 & c_3 \end{array}\right)$$

We find $[T]_{B',B}$ using an augmented matrix

that has new basis vectors as columns on the left and images of old basis vectors as columns on the right:

a) We can find $[T]_{B',B}$ as follows.

First, find $T(\vec{u}_1)$ and $T(\vec{u}_2)$.

$$T(\vec{u}_1) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1+2\cdot 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

$$T(\vec{u}_2) = T\left(\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -2+2\cdot 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 7 & 6 \\ 1 & 2 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row reducing this gives us representations of $\underline{T(\vec{u}_1)}$ and $\underline{T(\vec{u}_2)}$ in terms of B' vectors. That is, we'll have $[T(\vec{u}_1)]_{B'}$ and $[T(\vec{u}_2)]_{B'}$.

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 7 & 6 \\ 1 & 2 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 8/3 & 4/3 \end{array}\right]$$

$$[T]_{B',B} = \left[\, T(\vec{u}_1)_{B'} \;\middle|\; [T(\vec{u}_2)]_{B'} \,\right] = \boxed{\begin{bmatrix} 0 & 0 \\ -1/2 & 1 \\ 8/3 & 4/3 \end{bmatrix}}$$

This is the matrix for the transformation

T relative to the bases B and B'.
It acts on coordinate vectors relative to B and returns coordinate vectors relative to B'.

b. Verify that the formula $[T]_{B',B}\,[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$ holds for every vector in R^2.

LHS: • Find $[\vec{x}]_B$ for an arbitrary vector in R^2:

$$c_1\vec{u}_1 + c_2\vec{u}_2 = \begin{bmatrix}x_1\\x_2\end{bmatrix} \to \left[\begin{array}{cc|c}1&-2&x_1\\3&4&x_2\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}1&0&\frac{2x_1+x_2}{5}\\0&0&\frac{-3x_1+x_2}{10}\end{array}\right]$$

• Multiply

$$\begin{bmatrix}0&0\\-1/2&1\\8/3&4/3\end{bmatrix}\begin{bmatrix}\frac{2x_1+x_2}{5}\\\frac{-3x_1+x_2}{10}\end{bmatrix} = \begin{bmatrix}0\\-\frac{1}{2}x_1\\\frac{2}{3}x_1+\frac{2}{3}x_2\end{bmatrix}$$

RHS: • Find $[T(\vec{x})]_{B'}$ for $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2\\-x_1\\0\end{bmatrix} \to \left[\begin{array}{ccc|c}1&2&3&x_1+2x_2\\1&2&0&-x_1\\1&0&0&0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-\frac{1}{2}x_1\\0&0&1&\frac{2}{3}x_1+\frac{2}{3}x_2\end{array}\right]$$

Same

For a specific example, suppose we want use $[T]_{B',B}$ to find $T\left(\begin{bmatrix}11\\3\end{bmatrix}\right)$, where

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}x_1+2x_2\\-x_1\\0\end{bmatrix}$$

step 1: Express $\vec{x}=(11,3)$ in terms of B.

$$\left[\begin{array}{cc|c}1&-2&11\\3&4&3\end{array}\right]\to\left[\begin{array}{cc|c}1&0&5\\0&1&-3\end{array}\right],\text{ so}$$

$$[(11,3)]_B=(5,-3)$$

Step 2: multiply $\begin{bmatrix}0&0\\-1/2&1\\8/3&4/3\end{bmatrix}\begin{bmatrix}5\\-3\end{bmatrix}=\begin{bmatrix}0\\-11/2\\28/3\end{bmatrix}=[T(\vec{x})]_{B'}$,

Step 3: Use the new coord. vector w/ B'.

$$0\vec{v}_1 - \frac{11}{2}\vec{v}_2 + \frac{28}{3}\vec{v}_3 = -\frac{11}{2}\begin{bmatrix}2\\2\\0\end{bmatrix} + \frac{28}{3}\begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}17\\-11\\0\end{bmatrix} = T(\vec{x})$$

Recap: $(11,3) \to (5,-3) \to \left(0,-\frac{11}{2},\frac{28}{3}\right) \to (17,-11,0)$

$$\vec{x} \xrightarrow{\text{row reduce}} [\vec{x}]_B \xrightarrow{\text{Multiply}} [T(\vec{x})]_{B'} \xrightarrow{\text{lin comb.}} T(\vec{x})$$

Direct Check $T\left(\begin{bmatrix}11\\3\end{bmatrix}\right) = \begin{bmatrix}11+2\cdot 3\\-11\\0\end{bmatrix} = \begin{bmatrix}17\\-11\\0\end{bmatrix}$

We can visualize the process as follows:

$$\begin{tikzcd}
{[\vec{x}]_B = \begin{bmatrix}5\\-3\end{bmatrix}} \arrow[r, "②\ {[T]_{B',B}[\vec{x}]_B}"] & {[T(\vec{x})]_{B'} = \begin{bmatrix}0\\-\frac{11}{2}\\\frac{28}{3}\end{bmatrix}} \arrow[d, "③"] \\
{\vec{x} = \begin{bmatrix}11\\3\end{bmatrix}} \arrow[u, "①"] \arrow[r, "T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2\\-x_1\\0\end{bmatrix}", "\text{direct}"'] & {T(\vec{x}) = \begin{bmatrix}17\\-11\\0\end{bmatrix}}
\end{tikzcd}$$

Let V be an n-dimensional vector space with basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and let W be an m-dimensional vector space with basis B'. Suppose that $T : V \to W$ is a linear transformation, and that for each vector $\mathbf{x}$ in V, the coordinate vectors for $\mathbf{x}$ and $T(\mathbf{x})$ are $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_{B'}$, respectively. Then the **matrix for T relative to the bases B and B′** is written [by this author] as

$[T]_{B',B} = \left[[T(\mathbf{u}_1)]_{B'} \middle| [T(\mathbf{u}_2)]_{B'} \middle| \dots \middle| [T(\mathbf{u}_n)]_{B'}\right].$

This matrix has the property that $[T]_{B',B}\,[\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$.

#8 Let $T : P_2 \to P_3$ be the linear transformation defined by $T(p(x)) = xp(x-3)$, that is, $T\left(c_0 + c_1x + c_2x^2\right) = x\left(c_0 + c_1(x-3) + c_2(x-3)^2\right)$

a. Find $[T]_{B',B}$ relative to the bases $B = \left\{1,\ x,\ x^2\right\}$ and $B' = \left\{1,\ x,\ x^2, x^3\right\}$.

b. Use the three-step procedure illustrated in Example 2 to compute $T\left(1 + x - x^2\right)$.

c. Check the result obtained in part (b) by computing $T\left(1 + x - x^2\right)$ directly.

a. $\left[\begin{array}{c|c} \text{new basis vectors} & \text{images of old basis vectors} \end{array}\right]$ $\vec{P}_1 = 1,\ \vec{P}_2 = x,\ \vec{P}_3 = x^2$

$$T(\vec{P}_1) = x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\quad T(\vec{P}_2) = x^2 - 3x = \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{P}_3) = x^3 - 6x^2 + 9x = \begin{bmatrix} 0 \\ 9 \\ -6 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 1 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow [T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ 7 \\ -1 \end{bmatrix} \Rightarrow T(1+x-x^2) = -11x + 7x^2 - x^3$$

c) $$T(1+x-x^2) = x\left[1 + (x-3) - (x-3)^2\right] = x(1 + x - 3 - x^2 + 6x - 9)$$
$$= -11x + 7x^2 - x^3 \ \checkmark$$

In the special case where $T : V \to V$ is a linear operator with $B = B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, the matrix for T is called the **matrix for T relative to the basis B** and is written [by this author] as $[T]_B = \left[[T(\mathbf{u}_1)]_B \middle| [T(\mathbf{u}_2)]_B \middle| \ldots \middle| [T(\mathbf{u}_n)]_B\right]$.

#18 Let $D : P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.
a. Find the matrix for D relative to the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2 in which
$\mathbf{p}_1 = 2$, $\mathbf{p}_2 = 2 - 3x$, $\mathbf{p}_3 = 2 - 3x + 8x^2$. $\rightarrow\ -6+48x$
b. Use the matrix in part (a) to compute $D\left(6 - 6x + 24x^2\right)$.

a) $\left[\begin{array}{c|c} \text{new basis vectors} & \text{images of old basis vectors} \end{array}\right]$ $D(\vec{p}_1) = \frac{d}{dx}(2) = 0$,

$D(\vec{p}_2) = \frac{d}{dx}(2-3x) = -3$, $D(\vec{p}_3) = \frac{d}{dx}(2-3x+8x^2) = -3+16x$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & -3 & -3 \\ 0 & -3 & -3 & 0 & 0 & 16 \\ 0 & 0 & 8 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -3/2 & 23/6 \\ 0 & 1 & 0 & 0 & 0 & -16/3 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{[D]_B = \begin{bmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{bmatrix}}$$

b) $[D]_B\,\underline{[\vec{p}]_B}$: $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 6 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 8 & 24 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$$\begin{bmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \\ 0 \end{bmatrix} = [D(\vec{p})]_B$$

$$D(\vec{p}) = 13(2) - 16(2-3x) + 0(2-3x+8x^2)$$
$$= \boxed{-6+48x}$$

$$\begin{tikzcd} \overset{U}{\boxed{B}} \arrow[r] \arrow[rr, bend left] & \overset{V}{\boxed{B''}} \arrow[r] & \overset{W}{\boxed{B'}} \end{tikzcd}$$

Theorem 8.4.1 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if B, B'', and B' are bases for U, V, and W, respectively, then
$[T_2 \circ T_1]_{B',\, B} = [T_2]_{B',\, B''}\, [T_1]_{B'',\, B}$.

#12 Let $T_1 : P_1 \to P_2$ be the linear transformation defined by $T_1(p(x)) = xp(x)$ and let $T_2 : P_2 \to P_2$ be the linear operator defined by $T_2(p(x)) = p(2x+1)$. Let $B = \{1,\ x\}$ and $B' = \left\{1,\ x,\ x^2\right\}$ be the standard bases for P_1 and P_2.

a. Find $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B'}$, and $[T_1]_{B',B}$.

b. State a formula relating the matrices in part (a).

c. Verify that the matrices in part (a) satisfy the formula you stated in part (b).

a. For $[T_2]_{B'}$, find what T does to vectors in B' (no row reduction is necessary because B' — and B — are standard bases)

$T_2(1) = 1$, $T_2(x) = 2x+1$, $T_2(x^2) = 4x^2+4x+1$

$$[T_2]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}; \quad [T_1]_{B',B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b) $$\boxed{[T_2 \circ T_1]_{B',B} = [T_2]_B\,[T_1]_{B',B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 4 \end{bmatrix}$$

c) $$(T_2 \circ T_1)(a_0 + a_1 x) = T_2(a_0 x + a_1 x^2)$$
$$= a_0(2x+1) + a_1(2x+1)^2$$

$$= 2a_0x + a_0 + 4a_1x^2 + 4a_1x + a_1$$

$$= (a_0 + a_1,\ 2a_0 + 4a_1,\ 4a_1)$$

$$= \begin{bmatrix} a_0 + a_1 \\ 2a_0 + 4a_1 \\ 4a_1 \end{bmatrix}$$

Theorem 8.4.2

If $T : V \to V$ is a linear operator, and if B is a basis for V, then T is one-to-one if and only if $[T]_B$ is invertible. Moreover, when these conditions hold, $\left[T^{-1}\right]_B = [T]_B^{-1}$.